We can derive another recurrence for the exponential polynomials as follows. If α is an umbral operator, then

$$\alpha^*(A^k)' = \alpha^*(A^{k'})\,\alpha^*(A)'.$$

Applying to a polynomial $p(x)$ and using the properties of adjoints and derivations:

$$\langle A^k \mid \alpha x p(x)\rangle = \langle A^k \mid x\alpha(\mu(\alpha^*(A)')\,p(x))\rangle.$$

Therefore,

$$\alpha x p(x) = x\alpha(\mu(\alpha^*(A)')\,p(x)).$$

Now if we take α: $x^n \rightarrow \phi_n(x)$, then $\alpha^*(A) = e^A - \epsilon$ and so $\mu(\alpha^*(A)') = E^1$. Setting $p(x) = x^n$ gives

$$\phi_{n+1}(x) = x(\phi + 1)^n,$$

which, in terms of coefficients, gives the Stirling numbers recurrence

$$S(n+1, k) = \sum_{i \geqslant 0} \binom{n}{i} S(i, k-1).$$

9. Sheffer Sequences

So far, we have no explicit formula for shift-invariant operators. In obtaining an explicit formula for $\mu(L)$, we are led to a new class of polynomial sequences. A polynomial sequence $s_n(x)$ is a *Sheffer sequence relative to* a sequence $p_n(x)$ of binomial type if it satisfies the functional equation

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\,p_{n-k}(y)$$

for all $n \geqslant 0$ and for all $y \in K$.

Some characterizations of Sheffer sequences follow. The proofs follow a familiar pattern, and are therefore omitted.

Proposition 9.1. *A polynomial sequence $s_n(x)$ is a Sheffer sequence if and only if there exist a sequence of binomial type $p_n(x)$ and an invertible shift-invariant operator P such that*

$$p_n(x) = Ps_n(x)$$

for all $n \geqslant 0$.

PROPOSITION 9.2. *The following are equivalent for a polynomial sequence $s_n(x)$:*

(a) *The sequence $s_n(x)$ is a Sheffer sequence.*

(b) *There exists a delta operator Q such that*

$$Qs_n(x) = ns_{n-1}(x)$$

for all $n \geqslant 1$.

(c) *There exists a delta functional L and an invertible linear functional N such that*

$$\langle NL^k \mid s_n(x)\rangle = n!\, \delta_{n,k}$$

for all $n, k \geqslant 0$.

If Q is a delta operator and $Qs_n(x) = ns_{n-1}(x)$, we say that $s_n(x)$ is *Sheffer for Q*. Moreover, if $s_n(x)$ is a Sheffer sequence with respect to $p_n(x)$, the associated sequence for Q, then $s_n(x)$ is Sheffer for Q, and conversely. If T is an invertible shift-invariant operator, and $s_n(x)$ is Sheffer for Q, then $Ts_n(x)$ is also Sheffer for Q, and $T^n s_n(x)$ is Sheffer for $T^{-1}Q$.

Given a delta functional L and an invertible linear functional N, there exists exactly one polynomial sequence $s_n(x)$ satisfying

$$\langle NL^k \mid s_n(x)\rangle = n!\, \delta_{n,k},$$

namely, the sequence $s_n(x) = \mu(N)^{-1} p_n(x)$, where $p_n(x)$ is the associated sequence for L. We say that $s_n(x)$ is *the Sheffer sequence for N with respect to L*, or *the (N, L)-Sheffer sequence.*

A pair (N, L), where N is an invertible linear functional and L is a delta functional, determines a unique Sheffer sequence $s_n(x)$ in this way.

THEOREM 10 (Second Expansion Theorem). *Let $s_n(x)$ be the (N, L)-Sheffer sequence, and let $Q = \mu(L)$, $S = \mu(N)$. Then*

(a) *Every linear functional M can be uniquely expanded into the convergent series*

$$M = \sum_{k=0}^{\infty} \frac{\langle M \mid s_k(x)\rangle}{k!} L^k N.$$

(b) *Every shift-invariant operator T can be uniquely expanded into the convergent series*

$$T = \sum_{k=0}^{\infty} \frac{\langle \epsilon \mid Ts_k(x)\rangle}{k!} Q^k S.$$

(c) *Every polynomial $p(x)$ can be uniquely expanded into the finite sum*

$$p(x) = \sum_{k \geqslant 0} \frac{\langle NL^k \mid p(x)\rangle}{k!} s_k(x).$$

We can now give explicit formulas for shift-invariant operators:

THEOREM 11. (a) *Let $s_n(x)$ be a Sheffer sequence relative to the sequence $p_n(x)$ of binomial type. Every shift-invariant operator $\mu(L)$ can be represented by*

$$\mu(L)\, s_n(x) = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle\, s_{n-k}(x).$$

(b) *Conversely, suppose that for a delta operator $Q = \mu(L)$ there is a sequence of constants a_k such that*

$$Qs_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k s_{n-k}(x).$$

Then $s_n(x)$ is a Sheffer sequence relative to a sequence $p_n(x)$ of binomial type, and $a_n = \langle L \mid p_n(x)\rangle$ for all $n \geqslant 0$.

Proof. (a) Suppose $p_n(x)$ is the associated sequence for the delta functional M. Then

$$\begin{aligned}\mu(M^j)\, s_n(x) &= (n)_j\, s_{n-j}(x)\\ &= \sum_{k=0}^{n} \binom{n}{k} \langle M^j \mid p_k(x)\rangle\, s_{n-k}(x).\end{aligned}$$

By a closure argument, we may replace M^j by any linear functional. Q.E.D.

(b) Define the operator T by $Ts_n(x) = ns_{n-1}(x)$ for $n \geqslant 1$ and $Ts_0(x) = 0$. Then $s_n(x)$ will be a Sheffer sequence if T is shift-invariant. But

$$\begin{aligned}TQs_n(x) &= \sum_{k=0}^{n} \binom{n}{k} a_k Ts_{n-k}(x)\\ &= \sum_{k=0}^{n-1} \binom{n}{k} a_k(n-k)\, s_{n-k-1}(x)\\ &= n\sum_{k=0}^{n-1} \binom{n-1}{k} a_k s_{n-k-1}(x)\\ &= nQs_{n-1}(x) = QTs_n(x),\end{aligned}$$

and then Proposition 7.4 implies T is shift-invariant. Since $s_n(x)$ is a Sheffer sequence, part (a) implies $a_k = \langle L \mid p_k(x)\rangle$.

We define the *conjugate Sheffer sequence* of the pair (N, L) as the polynomial sequence

$$r_n(x) = \sum_{k\geqslant 0} \frac{\langle NL^k \mid x^n\rangle}{k!}\, x^k.$$

Not unexpectedly, it turns out that every conjugate Sheffer sequence is Sheffer, and conversely. The proofs of Proposition 9.3 and Theorem 12 below are similar to those of Proposition 4.4 and Theorem 4.

PROPOSITION 9.3. *A polynomial sequence*

$$s_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$$

is a Sheffer sequence with respect to the sequence of binomial type

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

if and only if

$$\binom{i+j}{i} c_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} c_{k,i} a_{n-k,j} . \qquad (***)$$

THEOREM 12. (a) *Every conjugate Sheffer sequence is a Sheffer sequence.*

(b) *Every Sheffer sequence is a conjugate Sheffer sequence.*

Every pair (N, L) is associated with two Sheffer sequences, its Sheffer sequence $s_n(x)$ and its conjugate Sheffer sequence $r_n(x)$. We say that $r_n(x)$ is *reciprocal Sheffer* to $s_n(x)$.

Similarly, a Sheffer sequence $s_n(x)$ is associated with two pairs, namely, the pair (M, L) for which $s_n(x)$ is the Sheffer sequence, and the pair $(\widetilde{M, L})$, for which $s_n(x)$ is the conjugate sequence. We say that $(\widetilde{M, L})$ is *the reciprocal pair* to (M, L).

Our goal now is to give a solution to the connection constants problem for Sheffer sequences. We proceed in a manner analogous to that for sequences of binomial type.

A *Sheffer operator* is a linear operator λ defined by λ: $x^n \to s_n(x)$, where $s_n(x)$ is a Sheffer sequence. If $Ps_n(x) = p_n(x)$, where $p_n(x)$ is of binomial type, then

$$\lambda = P^{-1} \circ \alpha$$

where α is the umbral operator α: $x^n \to p_n(x)$. We infer

THEOREM 13. *An operator λ is a Sheffer operator if and only if its adjoint is of the form $\beta \circ \mu(M^{-1})^*$, where β is a continuous automorphism of the umbral algebra, and $\mu(M^{-1})^*$ is multiplication by an invertible linear functional M^{-1}.*

Proof. If λ is a Sheffer operator then $\lambda = P^{-1} \circ \alpha$, where $P = \mu(M)$. Taking adjoints and applying Theorem 5 gives the result. The converse is obvious.

PROPOSITION 9.4. (a) *A Sheffer operator maps Sheffer sequences into Sheffer sequences.*

(b) *If λ: $s_n(x) \to r_n(x)$, where $r_n(x)$ is (N, R)-Sheffer and $s_n(x)$ is (M, L)-Sheffer, then λ is a Sheffer operator and* $\lambda^*(NR^k) = ML^k$, $k = 0, 1, 2,\ldots$.

We come now to the principal question for Sheffer sequences. Given two Sheffer sequences $r_n(x)$ and $s_n(x)$, determine the connection constants $c_{n,k}$ in

$$r_n(x) = \sum_{k=0}^{n} c_{n,k} s_k(x).$$

We know that the polynomial sequence

$$t_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$$

is also Sheffer. Thus the problem of computing the connection constants reduces to the problem of determining the pair of linear functionals which determine the sequence $t_n(x)$. Stated in other terms, the problem is to determine the pair of linear functionals corresponding to the umbral composition of two Sheffer sequences. We shall state the solution in terms of indicators.

PROPOSITION 9.5. *If the pair (M, L) with Sheffer sequence $s_n(x)$ has indicators $(f(t), g(t))$, and the pair (N, R) with Sheffer sequence $t_n(x)$ has indicators $(h(t), k(t))$, then the pair of the Sheffer sequence $s_n(\mathbf{t}(x))$ has indicators*

$$(f(t)\, h(g(t)),\, k(g(t))).$$

Proof. Let (X, Y) be the desired pair of linear functionals. Then clearly

$$\lambda^*_{(M,L)}: f(A)\, g(A)^k \to A^k,$$

$$\lambda^*_{(N,R)}: h(A)\, k(A)^k \to A^k,$$

and

$$\lambda^*_{(X,Y)}: XY^k \to A^k.$$

Therefore,

$$\begin{aligned} XY^k &= (\lambda^*_{(X,Y)})^{-1}(A^k) = (\lambda^*_{(M,L)})^{-1}(\lambda^*_{(N,R)})^{-1}(A^k) \\ &= (\lambda^*_{(M,L)})^{-1}(h(A)\, k(A)^k) \\ &= \mu(M)^*(\alpha_L{}^*)^{-1}(h(A)\, k(A)^k) \\ &= Mh((\alpha_L{}^*)^{-1} A)\, k((\alpha_L{}^*)^{-1} A)^k \\ &= Mh(L)\, k(L)^k \\ &= f(t)\, h(g(t))\, k(g(t))^k. \end{aligned}$$

Q.E.D.

COROLLARY 1. *If the pair (M, L) has indicators $(f(t), g(t))$, then the reciprocal pair $(\widetilde{M, L})$ has indicators*

$$\left(\frac{1}{f(g^{-1}(t))}, g^{-1}(t)\right).$$

COROLLARY 2. *Suppose $s_n(x)$ is Sheffer for (M, L), with indicators $(f(t), g(t))$ and $r_n(x)$ is Sheffer for (N, R), with indicators $(h(t), k(t))$. If*

$$r_n(x) = t_n(\mathbf{s}(x))$$

for a polynomial sequence $t_n(x)$, then $t_n(x)$ is Sheffer for the pair with indicators

$$\left(\frac{h(g^{-1}(t))}{f(g^{-1}(t))}, k(g^{-1}(t))\right).$$

The following theorem is a recurrence formula for Sheffer sequences.

THEOREM 14. *Let $s_n(x)$ be a Sheffer sequence relative to the associated sequence $p_n(x)$ for $Q = \mu(L)$, and let $Ps_n(x) = p_n(x)$. Then*

$$s_{n+1}(x) = (P\partial_Q(P^{-1}) + \theta_L)\, s_n(x).$$

Proof. First notice that

$$s_{n+1}(x) = P^{-1}p_{n+1}(x) = P^{-1}\theta_L p_n(x) = P^{-1}\theta_L P s_n(x).$$

From part (b) of Theorem 7, we have

$$P^{-1}\theta_L P = (P^{-1}\theta_L - \theta_L P^{-1})P + \theta_L = P\partial_Q(P^{-1}) + \theta_L,$$

hence the conclusion.

A wide variety of polynomial sequences studied since Euler turned out to be Sheffer sequences, and no computer list can be drawn here. We shall only give a few examples to illustrate how the seemingly endless variety of identities is in fact the repetition of a few general formulas.

Sheffer sequences relative to the sequence x^n are called *Appell sequences.* Some of the best-known instances are:

The *Bernoulli polynomials*, defined by the functional

$$\langle \gamma \mid p(x)\rangle = \int_0^1 p(x)\, dx.$$

Thus,

$$\langle \gamma A^k \mid B_n(x)\rangle = n!\, \delta_{kn},$$

or, setting $J = \mu(\gamma)$, in operator notation $B_n(x) = J^{-1}x^n$. All identities for Bernoulli polynomials follow from the definition and from the above theory.

For example, an application of the Second Expansion Theorem gives the Euler–MacLaurin expansion formula

$$\epsilon_a = \sum_{k\geqslant 0} \frac{\langle \epsilon_a \mid B_k(x)\rangle}{k!} \gamma A^k,$$

or more explicitly

$$p(x) = \sum_{k\geqslant 0} \frac{B_k(x)}{k!} \int_0^1 p^{(k)}(t)\, dt.$$

Similarly, the *Euler functional e* defined by

$$\langle e \mid p(x)\rangle = \frac{p(1) + p(0)}{2}$$

gives the *Euler polynomials* $e_n(x) = \mu(e)^{-1} x^n$ and again the Second Expansion Theorem delivers *Boole's summation formula*

$$p(x) = \sum_{k\geqslant 0} \frac{e_k(x)}{k!} \langle \epsilon \mid p^{(k)}(x)\rangle.$$

Along the same lines, the *Boole polynomials* are the Sheffer set $\zeta_n(x) = \mu(e)^{-1}(x)_n$, and the corresponding expansion goes by the name of *Boole's second summation formula*

$$p(x) = \sum_{k\geqslant 0} \frac{\zeta_k(x)}{k!} \langle e \mid \Delta^k p(x)\rangle.$$

The *Bernoulli polynomials of the second kind* are the Sheffer sequence defined by $b_n(x) = J(x)_n$, so that, for example, the identity

$$b_n(0) = \sum_{k=0}^{n} s(n, k)/(k + 1)$$

is trivial in the present context. The corresponding expansion gives a variant of the Euler–MacLaurin formula where derivatives are replaced by differences.

The umbral composition of Appell sequences reduces to the following simple rule: The Appell sequence $r_n(x) = t_n(\mathbf{s}(x))$ is the sequence TSx^n, where $t_n(x) = Tx^n$ and $s_n(x) = Sx^n$.

10. Factor Sequences

An *inverse formal power series*—or inverse series for short—is a formal power series of the form

$$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots = \sum_{k=1}^{\infty} a_k x^{-k}.$$

The family Γ of all such formal power series is an algebra under ordinary addition, formal multiplication and multiplication by scalars; the algebra does not have an identity. The series $f(x)$ is said to be of degree $-n$ if $a_1 = a_2 = \cdots = a_{n-1} = 0$, but $a_n \neq 0$.

In a *sequence* $f_{-n}(x)$ of inverse formal power series it is tacitly understood that $f_{-n}(x)$ is of degree $-n$, for $n = 1, 2, \ldots$.

We indicate sequences of inverse formal power series by the notation $f_{-n}(x)$, $n = 1, 2, \ldots$, in contrast to polynomial sequences $p_n(x)$. We endow Γ with a topology which stipulates that a sequence $f_{-n}(x) = \sum_{k=1}^{\infty} a_{n,k}x^{-k}$ converges to $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$ if, for each k, there exists an index n_k such that if $n > n_k$ then $a_{n,k} = a_k$. Under this topology, Γ becomes a topological algebra, and every sequence $f_{-n}(x)$ spans; that is, every inverse formal power series $f(x)$ can be uniquely expressed as a convergent series $f(x) = \sum_{k\geq 1} a_k f_{-k}(x)$ for suitable constants a_k.

Recalling that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}; \qquad \binom{0}{k} = \delta_{0,k},$$

for a scalar a, we set

$$(x + a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k x^{-n-k},$$

the right-hand series being convergent. One easily verifies that

$$(x + a)^{-m}(x + a)^{-n} = (x + a)^{-m-n}.$$

The symmetry in x and a of the left side is deceptive. The variable a ranges over all scalars, but x is not a variable at all, unlike the case of polynomials. Unlike polynomials, one may not "evaluate" an inverse formal power series by giving x a constant value.

For any inverse formal power series $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$, we may define $f(x + a)$ as

$$E^a: f(x) = \sum_{k=1}^{\infty} a_k(x + a)^{-k},$$

since the series on the right converges. The resulting operator E^a is again called the *translation operator.*

The *derivative* operator D on the algebra Γ is defined by setting $Dx^{-n} = -nx^{-n-1}$ and extending to all of Γ by closure.

We introduce the notion of *factor sequence*, which is in some ways analogous to a Sheffer sequence. Let $f_{-n}(x)$, $n = 1, 2, \ldots$, be a sequence of inverse formal power series, where $f_{-n}(x)$ is of degree $-n$. We say $f_{-n}(x)$ is a *factor sequence relative to* the sequence $p_n(x)$ of binomial type if it satisfies:

$$f_{-n}(x+a) = \sum_{k=0}^{\infty} \binom{-n}{k} p_k(a) f_{-n-k}(x), \qquad (*)$$

for all $n = 1, 2,\ldots$ and for all scalars $a \in K$. The identity $(*)$ is called the *factor* (*binomial*) *identity*. If $p_n(x)$ is the associated sequence for the delta functional L, we say that $f_{-n}(x)$ is the *factor sequence associated to* the delta functional L. Caution: again the symbols x and a cannot be interchanged in $(*)$.

The simplest factor sequence is the sequence x^{-n}, $n = 1, 2,\ldots$, which satisfies the factor (binomial) identity:

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k x^{-n-k}.$$

Our first goal is to establish an algebra isomorphism from the umbral algebra P^* into the algebra of linear operators on Γ. For any linear functional $L \in P^*$, we define the linear operator $\sigma(L)$, mapping Γ into Γ by

$$\sigma(L)\, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k}. \qquad (**)$$

We must show that $\sigma(L)$ can be defined on all inverse formal power series. To this end, if $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$, set $\sigma(L) f(x) = \sum_{k=1}^{\infty} a_k \sigma(L)\, x^{-k}$. Since the degree of $\sigma(L)\, x^{-k}$ is at most $-k$, this series is convergent. In other words, we may extend definition $(**)$ by closure to all of Γ. Thus, $\sigma(L)$ is a continuous operator on Γ.

The dual space Γ^* to Γ, that is, the vector space of all continuous linear functionals on Γ, is easily described. It consists of all linear functionals M on Γ such that $\langle M \mid x^{-n} \rangle = 0$ for all nonnegative integers n, except for a finite number.

Now consider $\sigma(L)^*$, the adjoint of the linear operator $\sigma(L)$, acting on the dual space Γ^*.

For any continuous linear functional M in Γ^*, we have

$$\begin{aligned} \langle \sigma(L)^* M \mid x^{-n} \rangle &= \langle M \mid \sigma(L)\, x^{-n} \rangle \\ &= \left\langle M \,\middle|\, \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k} \right\rangle \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle \langle M \mid x^{-n-k} \rangle. \qquad (***) \end{aligned}$$

Moreover, in $(***)$, the sequence x^{-n} can be replaced by an arbitrary factor sequence $f_{-n}(x)$:

THEOREM 15. *Let L be a linear functional in P^* and let $f_{-n}(x)$ be a factor sequence relative to the sequence $p_n(x)$. Then, for any continuous linear functional M in Γ^*, we have*

$$\langle \sigma(L)^* M \mid f_{-n}(x)\rangle = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle\langle M \mid f_{-n-k}(x)\rangle.$$

Proof. Let $\Gamma(x, y)$ be the topological algebra of all inverse formal power series in the variable x, whose coefficients are polynomials in the variable y. Define the map $L_y M_x$ of $\Gamma(x, y)$ into the field K by

$$L_y M_x(y^j x^{-i}) = \langle L \mid x^j\rangle\langle M \mid x^{-i}\rangle.$$

Since any element $f(x, y)$ in $\Gamma(x, y)$ is of the form $f(x, y) = \sum_{k=1}^{\infty} p_k(y)\, x^{-k}$, where $p_k(y)$ is a polynomial in y, and since $\langle M \mid x^{-k}\rangle = 0$ for all but a finite number of x^{-k}, we may define

$$L_y M_x f(x, y) = \sum_{k \geqslant 0} \langle L \mid p_k(x)\rangle\langle M \mid x^{-k}\rangle,$$

the sum on the right being finite. This makes $L_y M_x$ a continuous linear functional on $\Gamma(x, y)$. Thus equation $(***)$ becomes

$$\langle \sigma(L)^* M \mid x^{-n}\rangle = L_y M_x (x + y)^{-n}.$$

Since $\sigma(L)^* M$ is in Γ^*, it follows that $\langle \sigma(L)^* M \mid x^{-n}\rangle = 0$ except for a finite number of integers $n \geqslant 0$. For $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$ we have

$$\begin{aligned}
\langle \sigma(L)^* M \mid f(x)\rangle &= \left\langle \sigma(L)^* M \,\middle|\, \sum_{k=1}^{\infty} a_k x^{-k}\right\rangle \\
&= \sum_{k=1}^{\infty} a_k \langle \sigma(L)^* M \mid x^{-k}\rangle \\
&= \sum_{k=1}^{\infty} a_k L_y M_x (x + y)^{-k} \\
&= L_y M_x \sum_{k=1}^{\infty} a_k (x + y)^{-k} \\
&= L_y M_x f(x + y).
\end{aligned}$$

Finally, for the factor sequence $f_{-n}(x)$, we have

$$\begin{aligned}\langle\sigma(L)^* M \mid f_{-n}(x)\rangle &= L_y M_x f_{-n}(x+y)\\ &= L_y M_x \sum_{k=0}^{\infty} \binom{-n}{k} p_k(y) f_{-n-k}(x)\\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle\langle M \mid f_{-n-k}(x)\rangle. \qquad \text{Q.E.D.}\end{aligned}$$

An immediate corollary is a characterization of the shift invariant operators.

COROLLARY 1. *Let L be a linear functional in P^* and let $f_{-n}(x)$ be a factor sequence relative to the sequence $p_n(x)$. Then*

$$\sigma(L) f_{-n}(x) = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle f_{-n-k}(x).$$

PROPOSITION 10.1. *Let L and M be linear functionals in P^* and let N be a continuous linear functional in Γ^*. Then*

$$\sigma(L)^*(\sigma(M)^*N) = \sigma(LM)^*N.$$

Proof. On the one hand,

$$\begin{aligned}\langle\sigma(LM)^* N \mid x^{-n}\rangle &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle LM \mid x^k\rangle\langle N \mid x^{-n-k}\rangle\\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{-n}{k}\binom{k}{j} \langle L \mid x^j\rangle\langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\langle\sigma(L)^* (\sigma(M)^* N) \mid x^{-n}\rangle\\ &\quad= \sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle\langle\sigma(M)^* N \mid x^{-n-j}\rangle\\ &\quad= \sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle \sum_{i=0}^{\infty} \binom{-n-j}{i} \langle M \mid x^i\rangle\langle N \mid x^{-n-j-i}\rangle,\end{aligned}$$

and letting $k = i + j$, this equals

$$\begin{aligned}&\sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle \sum_{k=j}^{\infty} \binom{-n-j}{k-j} \langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle\\ &\quad= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{-n}{j}\binom{-n-j}{k-j} \langle L \mid x^j\rangle\langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle.\end{aligned}$$

We can now prove

PROPOSITION 10.2. *The map $L \to \sigma(L)$ is an algebra monomorphism from the umbral algebra P^* into the algebra of all continuous linear operators on Γ.*

Proof. We have already seen that $\sigma(L)$ is continuous. If $\sigma(L) = 0$, then

$$0 = \sigma(L)\, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k}$$

and thus $\langle L \mid x^k \rangle = 0$ for all $k \geqslant 0$, so that $L = 0$. Therefore σ is one-to-one. Finally, for any continuous N in Γ^* and any inverse formal power series $f(x)$ in Γ, we have

$$\begin{aligned}\langle N \mid \sigma(LM) f(x) \rangle &= \langle \sigma(LM)^* N \mid f(x) \rangle \\ &= \langle \sigma(L)^*(\sigma(M)^* N) \mid f(x) \rangle = \langle \sigma(M)^* N \mid \sigma(L) f(x) \rangle \\ &= \langle N \mid \sigma(M)\, \sigma(L) f(x) \rangle.\end{aligned}$$

Thus, $\sigma(LM) f(x) = \sigma(M)\, \sigma(L) f(x)$ and so $\sigma(ML) = \sigma(LM) = \sigma(M)\, \sigma(L)$ Therefore σ preserves multiplication and the proposition is proved.

We call the image of σ the algebra of *shift-invariant operators* on Γ, and denote this algebra by Ω.

COROLLARY 1. *A shift-invariant operator T in Ω is invertible if and only if $Tf(x)$ is of degree -1 whenever $f(x)$ is of degree -1.*

Let $p_n(x)$ be a sequence of binomial type, and let $f_{-1}(x)$ be the first member of a factor sequence, and thus of degree -1. If we choose an arbitrary sequence of constants c_k, $k = 0, 1, 2, \ldots$, and set

$$Tf_{-1}(x) = \sum_{k \geqslant 0} \binom{-1}{k} c_k f_{-1-k}(x).$$

then there exists a unique linear functional L in P^* such that $\langle L \mid p_k(x) \rangle = c_k$. Thus, setting

$$Tf_{-n}(x) = \sum_{k \geqslant 0} \binom{-n}{k} c_k f_{-n-k}(x)$$

we obtain a shift-invariant operator. In summary:

PROPOSITION 10.3. *Given a factor sequence $f_{-n}(x)$ and an inverse series $g(x)$, there is a unique shift-invariant operator T such that $Tf_{-1}(x) = g(x)$.*

We are now able to give the following characterization of the shift-invariant operators on Γ.

PROPOSITION 10.4. *A linear operator T on Γ is shift-invariant if and only if it is continuous and $TE^a = E^aT$, for all constants $a \in K$.*

Suppose T is a continuous operator on Γ with $TE^a = E^aT$ for all $a \in K$. Define constants c_k by

$$Tx^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} c_k x^{-1-k}.$$

By the previous proposition, there is a unique shift-invariant operator S for which Sx^{-1} and Tx^{-1}. Thus the operator $S - T$ is continuous, and satisfies $(S - T)E^a = E^a(S - T)$ and $(S - T)\,x^{-1} = 0$. Therefore we have

$$0 = E^a(S - T)\,x^{-1} = (S - T)\,E^a x^{-1}$$

$$= (S - T) \sum_{k=0}^{\infty} \binom{-1}{k} a^k x^{-1-k}$$

$$= \sum_{k=0}^{\infty} \binom{-1}{k} a^k (S - T)\,x^{-1-k}$$

for all $a \in K$. By alternatingly setting $a = 0$ and dividing by a we conclude that $(S - T)\,x^{-1-k} = 0$ for all $k \geqslant 0$. Thus $S = T$. Q.E.D.

We define a topology on the algebra of shift-invariant operators Ω as follows. A sequence T_m of operators converges to the operator T if given any inverse formal power series $f(x)$, and any continuous linear functional P in Γ^*, there exists an index m_0 such that if $m > m_0$ then $\langle P \mid T_m f(x)\rangle = \langle P \mid Tf(x)\rangle$. Under this topology, Ω is a topological algebra. Moreover, we have

PROPOSITION 10.5. *The isomorphism σ, mapping P^* onto Ω, is continuous.*

Proof. Let L_m be a sequence of linear functionals in P^*, with L_m converging to the zero functional. Then if P is a continuous linear functional in Γ^*, we have $\langle P \mid x^{-k}\rangle = 0$ for all but a finite number of exponents $k \geqslant 0$. Thus we may choose m_0 such that $m > m_0$ implies $\langle L_m \mid x^k\rangle = 0$ whenever $\langle P \mid x^{-k}\rangle \neq 0$. Then, if $m > m_0$,

$$\langle P \mid \sigma(L_m)\,x^{-n}\rangle = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L_m \mid x^k\rangle\langle P \mid x^{-n-k}\rangle$$

$$= 0$$

for all $n \geqslant 1$. Thus $\langle P \mid \sigma(L_m) f(x)\rangle = 0$ for all inverse formal power series $f(x)$.

We can now prove the Expansion Theorem for shift-invariant operators on Γ.

THEOREM 16 (Expansion Theorem). *Let $T = \sigma(M)$ be a shift-invariant operator, and let $Q = \sigma(L)$ be a delta operator, with associated sequence $p_n(x)$. Then*

$$T = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x)\rangle}{k!} Q^k.$$

Proof. The conclusion follows after applying the (continuous) isomorphism σ to the corresponding expansion of the linear functional M in powers of the delta functional L.

We call a shift-invariant operator Q a *delta operator* if $Q = \sigma(L)$ for some delta functional L.

PROPOSITION 10.6. *The sequence of inverse formal power series $f_{-n}(x)$, where the degree of $f_{-n}(x)$ is $-n$, is a factor sequence if and only if there exists a delta operator Q such that*

$$Qf_{-n}(x) = -nf_{-n-1}(x).$$

Proof. If $f_{-n}(x)$ is a factor sequence relative to the associated sequence $p_n(x)$ for the delta functional L, then

$$\begin{aligned}\sigma(L)f_{-n}(x) &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle f_{n-k}(x)\\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \delta_{k,1} f_{-n-k}(x)\\ &= -nf_{-n-1}(x).\end{aligned}$$

Conversely, if $Qf_{-n}(x) = -nf_{-n-1}(x)$, for some delta operator Q, then if $p_n(x)$ is the associated sequence for Q, by the Expansion Theorem,

$$E^a = \sum_{k=0}^{\infty} \frac{p_k(a)}{k!} Q^k$$

and hence

$$\begin{aligned}f_{-n}(x+a) &= \sum_{k=0}^{\infty} \frac{p_k(a)}{k!} Q^k f_{-n}(x)\\ &= \sum_{k=0}^{\infty} \binom{-n}{k} p_k(a) f_{-n-k}(x).\end{aligned}$$

Thus $f_{-n}(x)$ is a factor sequence.

COROLLARY 1. *Given any inverse formal power series $f_{-1}(x)$ of degree -1, and a sequence $p_n(x)$ of binomial type, there is a unique factor sequence $f_{-n}(x)$ for which $f_{-1}(x)$ is the first member.*

To preserve the analogy with Sheffer sequences, if Q is a delta operator, and $Qf_{-n}(x) = -nf_{-n-1}(x)$ we say that $f_{-n}(x)$ is a *factor sequence for Q*. Moreover, if $f_{-n}(x)$ is a factor sequence relative to $p_n(x)$, the associated sequence for $\sigma^{-1}(Q)$, then $f_{-n}(x)$ is a factor sequence for Q, and conversely. By the previous proposition, if T is an invertible shift-invariant operator and $f_{-n}(x)$ is a factor sequence for Q, then $Tf_{-n}(x)$ is also a factor sequence for Q, and $T^n f_{-n}(x)$ is a factor sequence for TQ.

Suppose $f_{-n}(x)$ and $h_{-n}(x)$ are factor sequences relative to the sequence $p_n(x)$ of binomial type. Then by Proposition 10.3, there exists a shift-invariant operator T for which $Tf_{-1}(x) = h_{-1}(x)$. But since $Tf_{-n}(x)$ is a factor sequence relative to $p_n(x)$, the previous corollary implies $Tf_{-n}(x) = h_{-n}(x)$. Thus any two factor sequences relative to the same sequence of binomial type are related by a shift-invariant operator.

The correspondence between linear functionals in the umbral algebra and the shift-invariant operators on Γ can be recast in a suggestive form as follows. Again we consider the algebra $\Gamma(x, y)$ of inverse formal power series in the variable x whose coefficients are polynomials in the variable y. If $T = \sigma(L)$ is a shift-invariant operator on Γ, we denote by the same letter T the operator $\mu(L)$, operating on the vector space of polynomials in the variable y. Then the identity

$$Tf_{-n}(x + a) = \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon_a \mid p_k(y) \rangle \, Tf_{-n-k}(x)$$

$$= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon_a L \mid p_k(y) \rangle f_{-k-n}(x)$$

can be suggestively rewritten in the form

$$\sum_{k=0}^{\infty} \binom{-n}{k} p_k(y) \, Tf_{-n-k}(x) = \sum_{k=0}^{\infty} \binom{-n}{k} \, Tp_k(y) f_{-n-k}(x).$$

In other words, the action of a shift-invariant operator on a factor sequence can be "transferred" to the corresponding sequence of binomial type.

Proposition 10.6 shows that there is a strong analogy between factor sequences and Sheffer sequences. It is natural to single out those factor sequences which are the analogs of sequences of binomial type. We are led to define the *associated factor sequence* for a delta operator Q as the unique factor sequence $f_{-n}(x)$ for $Q = \sigma(L)$ whose first term is

$$f_{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \langle L \mid x^{1+k} \rangle \, x^{-1-k}.$$

If we define the *derivative* of $Q = \sigma(L)$ to be $Q' = \sigma(\partial_A L)$, then since $\partial_A L$ is invertible, so is Q', and we have

$$f_{-1}(x) = Q' x^{-1}.$$

We come now to the explicit computation of associated factor sequences:

THEOREM 17 (Transfer Formulas). *Let $Q = DS$ be a delta operator on Γ. Then if $f_{-n}(x)$ is the associated factor sequence for Γ, we have*

(1) $f_{-n}(x) = Q' S^{n-1} x^{-n}$,

(2) $f_{-n}(x) = x S^n x^{-n-1}$.

Proof. (1) Let $g_{-n}(x) = Q' S^{n-1} x^{-n}$. Then $Q g_{-n}(x) = -n g_{-n-1}(x)$ and so by Proposition 10.6, $g_{-n}(x)$ is a factor sequence, relative to the associated sequence for $\sigma^{-1}(Q)$. Moreover $g_{-1}(x) = Q' x^{-1} = f_{-1}(x)$ and so by Corollary 1 to Proposition 10.6, $g_{-n}(x) = f_{-n}(x)$.

(2) Letting $\sigma(M) = S$, the following string of identities verifies the equivalence of the right-hand sides of (1) and (2), thus proving part (2),

$$\begin{aligned}
Q' S^{n-1} x^{-n} &= \sigma(\partial_A(AM)\, M^{n-1})\, x^{-n} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \partial_A(AM)\, M^{n-1} \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \langle (M + A\partial_A M)\, M^{n-1} \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \frac{n+k}{n} \langle M^n \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n-1}{k} \langle M^n \mid x^k \rangle\, x^{-n-k} \\
&= x\sigma(M^n)\, x^{-n-1} = x S^n x^{-n-1}.
\end{aligned}$$

COROLLARY 1. *Let $f_{-n}(x)$ be the associated factor sequence for the delta operator Q and let $g_{-n}(x)$ be the associated factor sequence for the delta operator $R = QP$, where P is invertible. Then*

(1) $g_{-n}(x) = R' P^{n-1} (Q')^{-1} f_{-n}(x)$,

(2) $g_{-n}(x) = x P^n x^{-1} f_{-n}(x)$.

Proof. Let $Q = DS$ and $R = DT$, where S and T are invertible operators, and $P = S^{-1}T$. To prove part (1), we observe that part (1) of Theorem 17 gives

$$S^{-n+1}(Q')^{-1} f_{-n}(x) = x^{-n} = T^{-n+1}(R')^{-1} g_{-n}(x).$$

The result follows by solving for $g_{-n}(x)$. Part (2) is proved in the same manner using part (2) of Theorem 17.

Since any two delta operators Q and R are related by $QP = R$ for some invertible shift-invariant operators, Corollary 1 relates any two associated factor sequences.

The following corollary is immediate from Theorem 17.

COROLLARY 2. *If $f_{-n}(x)$ is the associated factor sequence for the delta operator $Q = \sigma(L)$, and if $L = AM$, we have*

$$f_{-n}(x) = \sum_{k=0}^{\infty} \binom{-n-1}{k} \langle M^n \mid x^k \rangle\, x^{-n-k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!}\, x^{-n-k}.$$

The Transfer Formula allows us to compute explicitly the coefficients of a factor sequence.

COROLLARY 3. *Let $g_{-n}(x)$ be a factor sequence relative to the delta functional $L = AM$, and let $g_{-n}(x) = Tf_{-n}(x)$, where $f_{-n}(x)$ is the associated factor sequence for L and $T = \sigma(N)$. Then*

$$g_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle N(L^n)' \mid x^{n+k-1} \rangle}{n!}\, x^{-n-k}.$$

Proof. The Transfer Formula gives:

$$g_{-n}(x) = Tf_{-n}(x) = T\sigma(L'M^{n-1})\, x^{-n}$$

$$= \sigma(NL'M^{n-1})\, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle NL'M^{n-1} \mid x^k \rangle\, x^{-n-k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\langle N(L^n)' \mid x^{n+k-1} \rangle}{n!}\, x^{-n-k}.$$

We next derive a recurrence formula for the associated factor sequence:

COROLLARY 4 (Recurrence Formula). *If $f_{-n}(x)$ is the associated factor sequence for the delta operator Q, then*

$$f_{-n-1}(x) = Q'x^{-1}f_{-n}(x).$$

Proof. By the second formula in the preceding theorem we have

$$f_{-n}(x) = xS^n x^{-n-1}$$

and so

$$\underset{\sim}{Q}'x^{-1}f_{-n}(x) = \underset{\sim}{Q}'S^n x^{-n-1},$$

which, by the first formula, equals $f_{-n-1}(x)$.

Given an invertible shift-invariant operator T and a delta operator $\underset{\sim}{Q}$, with associated factor sequence $f_{-n}(x)$, we say that the factor sequence $g_{-n}(x) = Tf_{-n}(x)$ is the (T, Q)-*factor sequence*. Clearly, any such pair (T, Q) determines a unique factor sequence, and conversely. Notice that, in the theory of Sheffer sequences, the role of T is played by T^{-1}.

Now we derive a recurrence formula for factor sequences, which is the analog of Theorem 14.

If $f_{-n}(x)$ is the associated factor sequence for the delta functional $M = \sigma^{-1}(Q)$, the *shift* θ_Q, is the linear operator on Γ defined by $\theta_Q f_{-n}(x) = f_{-n+1}(x)$, for $n \geqslant 2$. Notice that θ_Q is not everywhere defined on Γ. Now if T is a shift-invariant operator on Γ, by the Expansion Theorem we may expand T in powers of Q, say $T = g(Q)$. It is straightforward to verify that on the algebra Γ' of inverse formal power series of degree at most -2, the operator $T\theta_Q - \theta_Q T$ satisfies

$$T\theta_Q - \theta_Q T = g'(Q).$$

Thus, on Γ', $T\theta_Q - \theta_Q T$ is shift-invariant and we denote it by $\partial_Q T$.

PROPOSITION 10.7. *Let $g_{-n}(x)$ be a (T, Q)-factor sequence, and let $f_{-n}(x)$ be the associated factor sequence for $\underset{\sim}{Q}$. Then*

$$g_{-n+1}(x) = (T^{-1}\partial_Q T + \theta_Q)\, g_{-n}(x),$$

for $n \geqslant 2$.

Proof. The result follows from

$$\begin{aligned} g_{-n+1}(x) &= Tf_{-n+1}(x) = T\theta_Q f_{-n}(x) \\ &= T\theta_Q T^{-1} g_{-n}(x) = ((T\theta_Q - \theta_Q T)T^{-1} + \theta_Q)\, g_{-n}(x) \\ &= (T^{-1}\partial_Q T + \theta_Q)\, g_{-n}(x). \end{aligned}$$

We can now study the *umbral composition* of two factor sequences, say $f_{-n}(x) = \sum_{k=n}^{\infty} c_{n,k} x^{-k}$ and $g_{-n}(x)$. The umbral composition $f_{-n}(\mathbf{g}(x))$ is the sequence

$$f_{-n}(\mathbf{g}(x)) = \sum_{k=n}^{\infty} c_{n,k} g_{-k}(x).$$

LEMMA 1. *If L and M are delta functionals in P^*, then*

$$\langle (M \circ L)^n \mid x^{n+i} \rangle = \sum_{k=0}^{i} \frac{1}{(n+k)!} \langle L^n \mid x^{n+k} \rangle \langle M^{n+k} \mid x^{n+i} \rangle.$$

Proof. Let $p_n(x)$ and $q_n(x)$ be the conjugate sequences for L and M, respectively. Then by Proposition 6.2, $M \circ L = \widetilde{\tilde{L} \circ \tilde{M}}$ is the conjugate sequence for $q_n(\mathbf{p}(x))$. This yields the following string of identities:

$$\begin{aligned}\sum_{l=0}^{n+i} \frac{\langle (M \circ L)^l \mid x^{n+i} \rangle}{l!} x^l &= q_{n+i}(\mathbf{p}(x)) \\ &= \sum_{j=0}^{n+i} \frac{\langle M^j \mid x^{n+i} \rangle}{j!} \sum_{l=0}^{j} \frac{\langle L^l \mid x^j \rangle}{l!} x^l \\ &= \sum_{l=0}^{n+i} \frac{1}{l!} \sum_{j=l}^{n+i} \frac{\langle M^j \mid x^{n+i} \rangle \langle L^l \mid x^j \rangle}{j!} x^l \\ &= \sum_{l=0}^{n+i} \frac{1}{l!} \sum_{k=0}^{n+i-l} \frac{\langle M^{k+l} \mid x^{n+i} \rangle \langle L^l \mid x^{k+l} \rangle}{(k+l)!} x^l.\end{aligned}$$

Comparing the coefficients of x^n in the first and last formula gives the result.

We can now prove

THEOREM 18. *If $f_{-n}(x)$ is the associated factor sequence for the delta operator $Q = \sigma(L)$ and if $g_{-n}(x)$ is the associated factor sequence for the delta operator $R = \sigma(M)$, then the umbral composition $f_{-n}(\mathbf{g}(x))$ is the associated factor sequence for the delta operator $\sigma(M \circ L)$.*

Proof. By Corollary 2 of Theorem 17 we have

$$f_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!} x^{-n-k}$$

and

$$g_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle M^n \mid x^{n+k} \rangle}{n!} x^{-n-k}.$$

Thus

$$\begin{aligned} f_{-n}(\mathbf{g}(x)) &= \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!} \sum_{j=0}^{\infty} (-1)^j \frac{\langle M^{n+k} \mid x^{n+k+j} \rangle}{(n+k)!} x^{-n-k-j} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{k+j} \frac{1}{n!\,(n+k)!} \langle L^n \mid x^{n+k} \rangle \langle M^{n+k} \mid x^{n+k+j} \rangle x^{-n-k-j} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{n!} \sum_{k=0}^{i} \frac{1}{(n+k)!} \langle L^n \quad x^{n+k} \rangle \langle M^{n+k} \mid x^{n+i} \rangle x^{-n-i}. \end{aligned}$$

By Lemma 1, the last member simplifies to

$$\sum_{i=0}^{\infty} (-1)^i \frac{\langle (M \circ L)^n \mid x^{n+i} \rangle}{n!} x^{-n-i},$$

which is the associated factor sequence for the delta functional $M \circ L$.

We may carry the analogy a step further with

THEOREM 19. *If $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence and if $g_{-n}(x)$ is the $(h(D), k(D))$-factor sequence, then the umbral composition $g_{-n}(\mathbf{f}(x))$ is the factor sequence with pair*

$$(f(D)\, h(g(D)), k(g(D))).$$

Proof. Suppose $g(D)$ has associated sequence $p_n(x)$ and associated factor sequence $h_{-n}(x)$, and suppose $k(D)$ has associated sequence $q_n(x)$ and associated factor sequence $k_{-n}(x)$. Then if we let $T = f(D)$ and $S = h(D)$,

$$\begin{aligned} g_{-n}(\mathbf{f}(x)) = Sk_{-n}(\mathbf{f}(x)) &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon \mid Sq_k(x) \rangle\, k_{-n-k}(\mathbf{f}(x)) \\ &= T \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon \mid Sq_k(x) \rangle\, k_{-n-k}(\mathbf{h}(x)). \end{aligned}$$

Defining the umbral operator $\alpha\colon x^n \to p_n(x)$, then $(\alpha^{-1})^* A = g(A)$ and so the above equals

$$\begin{aligned} T \sum_{k=0}^{\infty} &\binom{-n}{k} \langle (\alpha^{-1})^* \sigma^{-1}(S) \mid q_k(\mathbf{p}(x)) \rangle\, k_{-n-k}(\mathbf{h}(x)) \\ &= T\sigma((\alpha^{-1})^* \sigma^{-1}(S))\, k_{-n}(\mathbf{h}(x)) \\ &= f(D)\, h(g(D))\, k_n(\mathbf{h}(x)). \end{aligned}$$

The result follows.

We can now give a solution to the connection constants problem for factor sequences.

COROLLARY 1. *Suppose $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence and $g_{-n}(x)$ is the $(h(D), k(D))$-factor sequence, and suppose*

$$g_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} f_{-k}(x)$$

for constants $c_{-n,k}$. Then the sequence $r_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} x^{-k}$ is a factor sequence for the pair

$$\left(\frac{h(g^{-1}(D))}{f(g^{-1}(D))}, k(g^{-1}(D)) \right).$$

COROLLARY 2. *Suppose $f_{-n}(x)$ is the associated factor sequence for $f(D)$ and $g_{-n}(x)$ is the associated factor sequence for $g(D)$, and suppose*

$$g_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} f_{-k}(x),$$

for constants $c_{-n,k}$. Then the sequence $r_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} x^{-k}$ is the associated factor sequence for $g(f^{-1}(D))$.

11. APPLICATIONS TO FORMAL POWER SERIES

Given a formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

we can define a linear functional L in P^* by $\langle L \mid x^k \rangle = a_k$. We call L the *generating functional* of the sequence a_k. The series $f(t)$ is the indicator of the linear functional L and $L = f(A)$.

When $a_0 = 0$ and $a_1 \neq 0$ we call $f(t)$ a *delta series.* We have seen that the composition $f(g(t))$ is well defined when the constant coefficient of $g(t)$ vanishes, in particular when $g(t)$ is a delta series, and that

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle f(g(A)) \mid x^k \rangle}{k!} t^k. \qquad (*)$$

If $f(t)$ is the indicator of the delta functional L, we have seen (Corollary 1 to Theorem 6) that

$$f^{-1}(t) = \sum_{k=0}^{\infty} \frac{\langle \tilde{L} \mid x^k \rangle}{k!} t^k.$$

That is, the reciprocal series $f^{-1}(t)$ is the indicator of $\tilde{L}$, the reciprocal functional to L.

If $f(t)$ and $g(t)$ are the indicators of the delta functionals L and M, then Theorem 6 tells us that $f(g(t))$ is the indicator of the delta functional $f(g(A)) = M \circ L$, and $(*)$ becomes

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle M \circ L \mid x^k \rangle}{k!} t^k. \qquad (**)$$

The problem of determining the composition of formal power series is thus equivalent to the problem of determining the composition of delta functionals. It turns out that the latter can often be explicitly computed by the present methods, as we shall see.